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Critical constraint on inflationary magnetogenesis

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Abstract. Recently, there are several reports that the cosmic magnetic fields on Mpc scale in void region is larger than $\sim 10^{-15}\text{G}$ with an uncertainty of a few orders from the current blazar observations. On the other hand, in inflationary magnetogenesis models, additional primordial curvature perturbations are inevitably produced from iso-curvature perturbations due to generated electromagnetic fields. We explore such induced curvature perturbations in a model independent way and obtained a severe upper bound for the energy scale of inflation from the observed cosmic magnetic fields and the observed amplitude of the curvature perturbation, as $\rho_{\text{inf}}^{1/4} < 30\text{GeV} \times (B_{\text{obs}}/10^{-15}\text{G})^{-1}$ where B_{obs} is the strength of the magnetic field at present. Therefore, without a dedicated low energy inflation model or an additional amplification of magnetic fields after inflation, inflationary magnetogenesis on Mpc scale is generally incompatible with CMB observations.

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1 Introduction

It has been known for a long time that galaxies and galactic clusters have their own magnetic fields [1–4]. However, their origin is a big mystery of astronomy and cosmology [5–7]. Recently the generation mechanism of the magnetic fields in the universe attracts much attention because there are several reports that magnetic fields are found even in void regions. Such void magnetic fields could be detected by blazar observations [8–13] and it is reported that their strength is larger than $\sim 10^{-15}\text{G}$ with an uncertainty of a few orders. On the other hand, the upper bound on primordial magnetic fields could be also obtained from the cosmic microwave background (CMB) and the large scale structure (LSS) observations, and current upper bound is roughly given by 10^{-9}G (see, e.g. [14, 15], and references therein)¹. Therefore we know there exist the magnetic fields in the universe with the strength,²

$$10^{-15}\text{G} \lesssim B_{\text{obs}} \lesssim 10^{-9}\text{G}. \quad (1.1)$$

Nevertheless, their origin is still unknown and no successful quantitative model is established. If the magnetic fields are produced in the primordial universe, they can seed the observed galactic and cluster magnetic field [17, 18] as well as directly explain the void magnetic fields.

As one of the mechanism of generating such cosmic magnetic fields, “inflationary magnetogenesis” has been widely discussed. In the context of the inflationary magnetogenesis, large scale magnetic fields, as well as the primordial curvature perturbations, are basically generated from the quantum fluctuations. Although many models of the generation of magnetic fields during inflation are proposed so far [19–29], it is known that these inflationary magnetogenesis models suffer from several problems, namely the strong coupling problem [22, 30, 31], the backreaction problem [25, 30–32], the anisotropy problem [33, 34] and the curvature perturbation problem [35–40]. In particular, the curvature perturbation problem, where the primordial curvature perturbations which are induced from the generated electromagnetic

¹ Ref. [16] reported an updated constraint on a primordial magnetic field during big bang nucleosynthesis (BBN) as 10^{-6}G .

² The upper bound is irrelevant for magnetic fields which are produced after CMB photons are radiated.

fields during inflation should not exceed the observed value of CMB experiments, gives strong constraints on inflationary magnetogenesis models. For examples, in our previous paper [39], we have intensively studied the curvature perturbation problem by using a specific model, so-called the kinetic coupling model [20], and showed that the allowed strength of the produced magnetic fields is far weaker than the observational lower bound given by eq. (1.1). Ref. [38] have investigated the curvature perturbation problem specifying the time evolution of the magnetic fields during inflation as the power-law of the conformal time and showed limits of the amplitude of the present magnetic fields for the monomial and the hill-top inflation models with several reheating scenarios.

Although investigation of the constraint on inflationary magnetogenesis in model dependent ways is important, to discuss whether inflationary magnetogenesis is *really* possible or not, model independent arguments should be also necessary. As for such discussion, in ref. [35] the authors have put the lower bound on the inflation energy scale ρ_{inf} only by requiring the production of magnetic fields with the sufficient strength $B_{\text{obs}} \sim 10^{-15}\text{G}$, but they assumed that the dominant primordial curvature perturbation is generated during the single slow-roll inflation. In ref. [31], apart from the curvature perturbation problem, by requiring to escape from the strong coupling and the backreaction problems, the upper bound on ρ_{inf} has been put in model independent ways.

In this paper, we consider the curvature perturbation problem of inflationary magnetogenesis in a model independent way and we do not specify the dominant contribution of the primordial curvature perturbations. That is, our result could be also applied to the case where the dominant primordial curvature perturbation is sourced from a light scalar field other than inflaton. We focus on the existence of the electric fields due to the time evolution of the magnetic fields in the Friedmann-Lemaître-Robertson-Walker (FLRW) universe and we show that if one requires inflation magnetogenesis is responsible for the generation of the observed magnetic fields and assumes no additional amplification after inflation, the inflation energy scale is constrained by the curvature power spectrum \mathcal{P}_ζ as

$$\mathcal{P}_\zeta^{\text{obs}} > \mathcal{P}_\zeta^{\text{em}} \quad \Rightarrow \quad \rho_{\text{inf}}^{1/4} < 30\text{GeV} \times \left(\frac{p_B}{1\text{Mpc}^{-1}} \right)^{\frac{5}{4}} \left(\frac{B_{\text{obs}}}{10^{-15}\text{G}} \right)^{-1}, \quad (1.2)$$

where ρ_{inf} is the energy scale of inflation, $p_B > 1\text{Mpc}^{-1}$ is the peak wave number of the void magnetic field and B_{obs} is the magnetic field strength today. Therefore, our result indicates some tension between inflationary magnetogenesis and phenomenologies in the very early universe, e.g., genesis of the baryon or dark matter, where high energy physics are involved.

We also discuss a possible way out of our constraint. If strong magnetic fields are produced without amplifying electric fields, one could avoid our constraint. Such situation is apparently realized in a tree-level analysis of the so-called strong coupling regime of the kinetic coupling model [20, 22, 25]. However, since the coupling constant becomes huge in the model, a non-perturbative analysis beyond the tree-level is required to make the correct prediction [30]. Furthermore, an additional amplification or a non-adiabatic dilution of magnetic fields after inflation can relax our constraint. For example, if the inverse cascade works, the constraint is alleviated [53, 54].

The rest of paper is organized as follows. In section 2, we briefly review the current lower bound on the cosmic magnetic field from the blazar observations and outline how we constrain inflationary magnetogenesis in a model independent way. In section 3, we derive an expression of curvature perturbations induced by electromagnetic fields during inflation. In section 4, the constraint on inflationary magnetogenesis is obtained. Section 5 is devoted to

a summary and discussions. In appendix, we discuss the constraint without the assumption of the instantaneous reheating.

2 Basic ideas

In this section, we briefly review the observation of the void magnetic field and basis of our idea. In addition, we briefly explain our approach to obtain the model independent constraint.

2.1 Observation of the void magnetic field

Recently it has been reported that magnetic fields in void regions are indirectly detected by gamma-ray observations of blazars [8–13]. In such current blazar observations, the strength and the correlation length of the magnetic field are degenerated³ and hence, in the literature, the lower bound on the magnetic strength is obtained by assuming its correlation length. Note that if the correlation length is larger than $\sim 1\text{Mpc}$ which is roughly the mean free path of electrons and positrons in void regions, the lower bound does not depend on the correlation length. On the other hand, in case where the correlation length is smaller than $\sim 1\text{Mpc}$, due to the randomness of the distribution of the magnetic fields, the effect of the magnetic fields along the line of sight should be proportional to $(L/1\text{Mpc})^{1/2}$ where L is a correlation length. That is, the lower bound for the strength of the magnetic fields is proportional to $(L/1\text{Mpc})^{-1/2}$. As a result, the reported lower bound for the peak strength of the magnetic field is given by [10, 11]

$$B(\eta_{\text{now}}, p_B) \gtrsim 10^{-15} \text{G} \times \begin{cases} \left(\frac{p_B}{1\text{Mpc}^{-1}}\right)^{1/2} & (p_B > 1\text{Mpc}^{-1}) \\ 1 & (p_B < 1\text{Mpc}^{-1}) \end{cases}, \quad (2.1)$$

where $B(\eta_{\text{now}}, k)$ denotes the void magnetic field at present in Fourier space, p_B is its peak wave number. Note that $B(\eta_{\text{now}}, k)$ is assumed to have a peak at $k = p_B$ with a peak width $\Delta \ln k = \mathcal{O}(1)$ in accordance with the definite correlation length p_B^{-1} .⁴ In this paper, we focus on the case with $p_B \geq 1\text{Mpc}^{-1}$.

2.2 Basis of our idea

Let us discuss general properties of electromagnetic fields in the FLRW universe including the inflation era. In the FLRW universe, the Fourier transformed components of the electromagnetic fields are given in terms of the vector potential as

$$E_i(\eta, \mathbf{k}) = -a^{-2} \partial_\eta A_i(\eta, \mathbf{k}), \quad B_i(\eta, \mathbf{k}) = a^{-2} i \epsilon_{ijl} k_j A_l(\eta, \mathbf{k}), \quad (2.2)$$

in the radiation gauge. Here, a is the scale factor, k denotes wave number, η denotes conformal time and $A_i(\eta, \mathbf{k})$ is the vector potential in Fourier space. Note that B_i is proportional to a^{-2} and substantially decrease as the universe expands. For simple discussion about the strength of the electromagnetic fields, here we suppress the vector legs of E_i, B_i and A_i . A mathematically strict treatment including the vector legs will be shown in the following sections.

³In future observations, it is expected that this degeneracy will be resolved [41].

⁴A more rigorous treatment of the magnetic lower bound is developed in the appendix of ref. [31].

If the magnetic field is generated during inflation and it monotonically decreases by the adiabatic dilution after the inflation, the present lower bound $B(\eta_{\text{now}}, p_B) \gtrsim 10^{-15} \text{G}$ can be translated into the lower bound on the strength of the magnetic field at the end of inflation as

$$B(\eta_f, p_B) \gtrsim 10^{-15} \text{G} \left(\frac{a_{\text{now}}}{a_f} \right)^2 = 2 \times 10^{40} \text{G} \left(\frac{\rho_{\text{inf}}^{1/4}}{10^{15} \text{GeV}} \right)^2, \quad (2.3)$$

where subscript f denotes the end of inflation and the instantaneous reheating is assumed for simplicity. Therefore, to explain the observational lower bound by inflationary magnetogenesis, strong magnetic fields should be produced during inflation. However, the magnetic field also decreases rapidly during inflation because of the factor a^{-2} . To compensate the adiabatic dilution and produce the magnetic field effectively, at least the vector potential $A(\eta, p_B)$ must be amplified faster than a^2 as

$$A(\eta, p_B) \propto |\eta|^{-n} \quad (n > 2). \quad (2.4)$$

In such case where the vector potential evolves in time, from eq. (2.2) we can easily find that the amplitude of the electric field should be much larger than that of the magnetic field on super-horizon scales. From eqs. (2.2) and (2.4), we obtain

$$\left| \frac{E}{B} \right| = \left| \frac{n}{k\eta} \right| = ne^{N_k} \gg 1 \quad (\text{on super-horizon scales}), \quad (2.5)$$

where $N_k \equiv -\ln |k\eta|$ is the e-fold number measured from the end of inflation to the time at the horizon exit of the k mode. This equation means that at the end of inflation the electric field is bigger than the magnetic field whose strength is eq. (2.3) by the factor of $ne^{N_{p_B}}$. Hence it is easy to imagine that including the effect of such strong electric field into the investigation of the inflationary magnetogenesis would give a strong constraint on the scenarios.

2.3 Model independent approach

While most previous works specify a model of magnetogenesis and fix the behavior of the vector potential $A(\eta, k)$, we assume $A(\eta, k)$ is well approximated by a power-law of η only for the last one e-fold of inflation. It should be noted that the vector potential $A(\eta)$ can be a more complicated function of η in general. In such case, the approximation of the simple power-law gets worse for considering long duration. However, in terms of obtaining a conservative constraint in model independent approach, it should be sufficient to focus on the contribution from the last one e-fold before the end of inflation and assume constant n during such short duration. We also consider only the contribution from the electromagnetic fields around the peak scale $k \sim p_B$ as shown in (2.1). Of course, in general the electromagnetic fields might have the power at the separate scales from the peak with depending on the models and they also give some contributions. Also in this respect, our constraint should be conservative, which is obtained in model independent approach. Thus, the key assumption of this paper for the vector potential is given by

$$A(\eta, k) = \left(\frac{\eta}{\eta_f} \right)^{-n} A(\eta_f, k), \quad \text{for } e\eta_f \leq \eta \leq \eta_f, \quad k \sim p_B, \quad \text{and } n = \text{const.} \quad (2.6)$$

By using this assumption for the vector potential, we will calculate the curvature perturbation induced by the electric field for the last one e-folding time and obtain the constraint by

requiring that the induced curvature perturbation is smaller than the observed value as eq. (1.2).

Before closing this section, it should be noted that the constraint apparently becomes very weak when $A(\eta, p_B)$ significantly grows before $N = 1$ and $A(\eta, p_B)$ is nearly constant, $|n| \ll 1$, for the last one e-fold. However, in that case, we can obtain an even more stringent constraint by considering not last one e-fold but the time when $n \sim \mathcal{O}(1)$ before the last one e-fold. The details of this case will be discussed in last part of section 4.

3 Power spectrum of Induced Curvature Perturbations

In this section, we derive an equation that evaluates the power spectrum of the curvature perturbation induced by the electric field during inflation.

It has been well known that the curvature perturbation on the uniform energy density hypersurface, ζ , should be constant in time on super-horizon scales when any iso-curvature component does not exist. In case that the electromagnetic fields generated during inflation behave as the iso-curvature perturbations, the evolution of the curvature perturbation ζ on super-horizon scales is given by [35, 40]⁵

$$\dot{\zeta}(t, \mathbf{x}) = -\frac{H(t)\delta p_{\text{nad}}(t, \mathbf{x})}{\rho(t) + p(t)}, \quad (3.1)$$

with the non-adiabatic pressure $\delta p_{\text{nad}}(t, \mathbf{x}) \equiv \delta p(t, \mathbf{x}) - \frac{\dot{p}(t)}{\rho(t)}\delta\rho(t, \mathbf{x})$ approximately given by

$$\delta p_{\text{nad}}(t, \mathbf{x}) \simeq \frac{4}{3}\rho^{\text{em}}(t, \mathbf{x}). \quad (3.2)$$

Here, H, ρ, p are the Hubble parameter, total energy density and pressure, respectively, superscript “em” denotes that a quantity is of the electromagnetic fields and $\rho^{\text{em}} = 3p^{\text{em}}$ is used. The anisotropic stress also contributes as a source term but we conservatively ignore it [35]. Integrating eq. (3.1), we obtain the Fourier transformed component of the curvature perturbations induced from the electromagnetic field as

$$\zeta_{\mathbf{k}}^{\text{em}} = 2 \int dN \frac{\rho_{\mathbf{k}}^{\text{em}}}{\epsilon \rho_{\text{inf}}}, \quad (3.3)$$

where subscript “inf” denotes that a quantity is of the inflaton, respectively.⁶ $N \equiv -\ln(a/a_f)$ is the e-folding number, and $\epsilon \equiv -\dot{H}/H^2$ is the slow-roll parameter.

The energy density of the electromagnetic field in Fourier space $\rho_{\mathbf{k}}^{\text{em}}$ is given by

$$\rho_{\mathbf{k}}^{\text{em}} = \frac{1}{2} \int \frac{d^3p d^3q}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{q} - \mathbf{k}) [\mathbf{E}(\eta, \mathbf{p}) \cdot \mathbf{E}(\eta, \mathbf{q}) + \mathbf{B}(\eta, \mathbf{p}) \cdot \mathbf{B}(\eta, \mathbf{q})]. \quad (3.4)$$

Note since $\rho^{\text{em}} = (\mathbf{E}^2 + \mathbf{B}^2)/2$ in the real space, $\rho_{\mathbf{k}}^{\text{em}}$ is written in terms of the convolution of the electromagnetic fields. In this paper, the kinetic term of the Maxwell theory, $\mathcal{L} = -F^{\mu\nu}F_{\mu\nu}/4$, is assumed. If one consider the kinetic coupling model where an arbitrary

⁵See also earlier intensive works [42–45].

⁶In the next leading order of the slow-roll parameter, the integrand in eq. (3.3) is multiplied by $(1 - \epsilon/2)$. This is because $p_{\text{inf}} = -\rho_{\text{inf}}(1 - 2\epsilon/3)$ and $\dot{p}_{\text{inf}}/\rho_{\text{inf}} \simeq -1 + 2\epsilon/3 + \mathcal{O}(\dot{\epsilon})$ in δp_{nad} . However, since this factor does not change the order of magnitude of the integral, we ignore it.

function of time $I(\eta)$ is multiplied, $\mathcal{L} = -I(\eta)F^{\mu\nu}F_{\mu\nu}/4$, eq. (3.4) is also multiplied by $I(\eta)$ (The relation between E and B given by eq. (2.5) still holds.). In such case, to avoid the strong coupling problem, $I(\eta)$ should be larger than unity even during inflation. Therefore, $\rho_{\mathbf{k}}^{\text{em}}$ is larger than eq. (3.4) and the resultant constraint becomes tighter. In other words, eq. (3.4) is a conservative estimate in view of the kinetic coupling model. Moreover, in inflationary magnetogenesis models, some interaction terms between A_μ and other fields (e.g. $\mathcal{L}_{\text{int}} = \frac{\phi}{M}F_{\mu\nu}\tilde{F}^{\mu\nu}$, where ϕ is a pseudo-scalar field [21, 27]) are considered to amplify the magnetic field. In those cases, the energy density of the interaction terms also contribute to source ζ . Nonetheless they can be conservatively ignored.⁷

In FLRW universe, the power spectra of the electromagnetic fields are respectively defined as⁸

$$\langle E_i(\eta, \mathbf{k}) E_j(\eta, \mathbf{k}') \rangle \equiv (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{1}{2} [\delta_{ij} - \hat{k}_i \hat{k}_j] \frac{2\pi^2}{k^3} \mathcal{P}_E(\eta, k), \quad (3.5)$$

$$\langle B_i(\eta, \mathbf{k}) B_j(\eta, \mathbf{k}') \rangle \equiv (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{1}{2} [\delta_{ij} - \hat{k}_i \hat{k}_j] \frac{2\pi^2}{k^3} \mathcal{P}_B(\eta, k), \quad (3.6)$$

where $\langle \dots \rangle$ denotes the vacuum expectation value. In the radiation gauge, the vector potential $A_i(\eta, \mathbf{k})$ is quantized as

$$A_i(\eta, \mathbf{k}) = \sum_{\lambda=1}^2 \epsilon_i^{(\lambda)}(\hat{\mathbf{k}}) \left[a_{\mathbf{k}}^{(\lambda)} \mathcal{A}_k(\eta) + a_{-\mathbf{k}}^{\dagger(\lambda)} \mathcal{A}_k^*(\eta) \right], \quad (3.7)$$

where $\epsilon_i^{(\lambda)}$ is the polarization vector, $a_{\mathbf{k}}^{\dagger(\lambda)}$ and $a_{\mathbf{k}}^{(\lambda)}$ are respectively creation and annihilation operators, a hat of $\hat{\mathbf{k}}$ denotes the unit vector and λ is a polarization label. The polarization vector $\epsilon_i^{(\lambda)}$ satisfies $k_i \epsilon_i^{(\lambda)}(\hat{\mathbf{k}}) = 0$, and $\sum_{\lambda=1}^2 \epsilon_i^{(\lambda)}(\hat{\mathbf{k}}) \epsilon_j^{(\lambda)}(-\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ while the creation/annihilation operators satisfy a commutation relation: $[a_{\mathbf{k}}^{(\lambda)}, a_{-\mathbf{k}'}^{\dagger(\sigma)}] = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \delta^{\lambda\sigma}$, as usual. From eqs. (2.2) and (3.7), the power spectra of the electromagnetic fields can be written in terms of the mode function of the vector potential, \mathcal{A}_k , as

$$\mathcal{P}_E(\eta, k) = \frac{k^3 |\partial_\eta \mathcal{A}_k|^2}{\pi^2 a^4}, \quad \mathcal{P}_B(\eta, k) = \frac{k^5 |\mathcal{A}_k|^2}{\pi^2 a^4}. \quad (3.8)$$

By using those equations, one can calculate the power spectrum of the curvature perturbation induced from the electromagnetic fields.

First, substituting eq. (3.4) into eq. (3.3), we obtain

$$\begin{aligned} \langle \zeta_{\mathbf{k}}^{\text{em}} \zeta_{\mathbf{k}'}^{\text{em}} \rangle &= \int dN dN' \frac{1}{\epsilon \rho_{\text{inf}}} \frac{1}{\epsilon \rho_{\text{inf}}} \int \frac{d^3 p d^3 q d^3 p' d^3 q'}{(2\pi)^6} \delta(\mathbf{p} + \mathbf{q} - \mathbf{k}) \delta(\mathbf{p}' + \mathbf{q}' - \mathbf{k}') \\ &\quad \times \langle (\mathbf{E}_{\mathbf{p}} \cdot \mathbf{E}_{\mathbf{q}} + \mathbf{B}_{\mathbf{p}} \cdot \mathbf{B}_{\mathbf{q}}) (\mathbf{E}_{\mathbf{p}'} \cdot \mathbf{E}_{\mathbf{q}'} + \mathbf{B}_{\mathbf{p}'} \cdot \mathbf{B}_{\mathbf{q}'} \rangle. \end{aligned} \quad (3.9)$$

⁷Only if the energy density of the interaction term is negative and it cancels the kinetic energy, our estimation becomes invalid. But no mechanism that leads to such cancellation is found [31].

⁸We consider the non-helical case where the parity is not violated. The extension to the helical case is straightforward [47].

Here, 4-point correlation functions of the electromagnetic fields appear. Then the 4-point correlation function of \mathbf{E} can be computed as ⁹

$$\begin{aligned} \langle \mathbf{E}_{\mathbf{p}} \cdot \mathbf{E}_{\mathbf{q}} \mathbf{E}_{\mathbf{p}'} \cdot \mathbf{E}_{\mathbf{q}'} \rangle &= a^{-4}(\eta) a^{-4}(\eta') \sum_{\lambda, \sigma, \lambda', \sigma'} \epsilon_i^{(\lambda)}(\hat{\mathbf{p}}) \epsilon_i^{(\sigma)}(\hat{\mathbf{q}}) \epsilon_j^{(\lambda')}(\hat{\mathbf{p}}') \epsilon_j^{(\sigma')}(\hat{\mathbf{q}}') \\ &\quad \times \partial_\eta \mathcal{A}_p(\eta) \partial_\eta \mathcal{A}_q(\eta) \partial_{\eta'} \mathcal{A}_{p'}(\eta') \partial_{\eta'} \mathcal{A}_{q'}(\eta') \\ &\quad \times \left\langle \left(a_{\mathbf{p}}^{(\lambda)} + a_{-\mathbf{p}}^{\dagger(\lambda)} \right) \left(a_{\mathbf{q}}^{(\sigma)} + a_{-\mathbf{q}}^{\dagger(\sigma)} \right) \left(a_{\mathbf{p}'}^{(\lambda')} + a_{-\mathbf{p}'}^{\dagger(\lambda')} \right) \left(a_{\mathbf{q}'}^{(\sigma')} + a_{-\mathbf{q}'}^{\dagger(\sigma')} \right) \right\rangle. \end{aligned} \quad (3.10)$$

Since the bracket of the annihilation/creation operators yields $2(2\pi)^6 \delta(\mathbf{p} + \mathbf{q}') \delta(\mathbf{p}' + \mathbf{q}) \delta^{\lambda\sigma'} \delta^{\lambda'\sigma}$ [39], performing the q and q' integrals by using the delta functions, one obtains

$$\begin{aligned} \int \frac{d^3 q d^3 q'}{(2\pi)^6} \langle \mathbf{E}_{\mathbf{p}} \cdot \mathbf{E}_{\mathbf{q}} \mathbf{E}_{\mathbf{p}'} \cdot \mathbf{E}_{\mathbf{q}'} \rangle \\ = 2a^{-4}(\eta) a^{-4}(\eta') \partial_\eta \mathcal{A}_p(\eta) \partial_\eta \mathcal{A}_{p'}^*(\eta) \partial_{\eta'} \mathcal{A}_{p'}(\eta') \partial_{\eta'} \mathcal{A}_p^*(\eta') \left[1 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')^2 \right]. \end{aligned} \quad (3.11)$$

Repeating similar calculations, one can show

$$\begin{aligned} \int \frac{d^3 q d^3 q'}{(2\pi)^6} \langle \mathbf{E}_{\mathbf{p}} \cdot \mathbf{E}_{\mathbf{q}} \mathbf{B}_{\mathbf{p}'} \cdot \mathbf{B}_{\mathbf{q}'} \rangle \\ = 4a^{-4}(\eta) a^{-4}(\eta') \partial_\eta \mathcal{A}_p(\eta) \partial_\eta \mathcal{A}_{p'}^*(\eta) \mathcal{A}_{p'}(\eta') \mathcal{A}_p^*(\eta') [\mathbf{p} \cdot \mathbf{p}']^2, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \int \frac{d^3 q d^3 q'}{(2\pi)^6} \langle \mathbf{B}_{\mathbf{p}} \cdot \mathbf{B}_{\mathbf{q}} \mathbf{B}_{\mathbf{p}'} \cdot \mathbf{B}_{\mathbf{q}'} \rangle \\ = 2a^{-4}(\eta) a^{-4}(\eta') \mathcal{A}_p(\eta) \mathcal{A}_{p'}^*(\eta) \mathcal{A}_{p'}(\eta') \mathcal{A}_p^*(\eta') p^2 p'^2 \left[1 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')^2 \right]. \end{aligned} \quad (3.13)$$

As we discussed in sec. 2.2, the magnetic field is far smaller than the electric field on super-horizon. Thus we neglect the contributions that include \mathbf{B} , namely eqs. (3.12) and (3.13), and focus on eq. (3.11). Note that this procedure underestimates eq. (3.9). Substituting eq. (3.11) into eq. (3.9), we obtain

$$\begin{aligned} \langle \zeta_{\mathbf{k}}^{\text{em}} \zeta_{\mathbf{k}'}^{\text{em}} \rangle &> 2\delta(\mathbf{k} + \mathbf{k}') \int dN dN' \frac{1}{\epsilon \rho_{\text{inf}}} \frac{1}{\epsilon \rho_{\text{inf}}} \int d^3 p d^3 p' \delta(\mathbf{p} - \mathbf{p}' - \mathbf{k}) \\ &\quad \frac{\partial_\eta \mathcal{A}_p(\eta) \partial_\eta \mathcal{A}_{p'}^*(\eta)}{a^4(\eta)} \frac{\partial_{\eta'} \mathcal{A}_{p'}(\eta') \partial_{\eta'} \mathcal{A}_p^*(\eta')}{a^4(\eta')} \left[1 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')^2 \right]. \end{aligned} \quad (3.14)$$

By using the definition of the curvature power spectrum given as

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_\zeta(k), \quad (3.15)$$

eq. (3.14) can be rewritten in terms of the power spectrum as

$$\begin{aligned} \mathcal{P}_\zeta^{\text{em}}(k) &> \frac{k^3}{2^3 \pi^5} \int dN dN' \frac{1}{\epsilon \rho_{\text{inf}}} \frac{1}{\epsilon \rho_{\text{inf}}} \int d^3 p d^3 p' \delta(\mathbf{p} - \mathbf{p}' - \mathbf{k}) \\ &\quad \frac{\partial_\eta \mathcal{A}_p(\eta) \partial_\eta \mathcal{A}_{p'}^*(\eta)}{a^4(\eta)} \frac{\partial_{\eta'} \mathcal{A}_{p'}(\eta') \partial_{\eta'} \mathcal{A}_p^*(\eta')}{a^4(\eta')} \left[1 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')^2 \right]. \end{aligned} \quad (3.16)$$

⁹Although the complex conjugations of \mathcal{A}_p are ignored in eq. (3.10) for simplicity, they should be included like $\left(a_{\mathbf{p}}^{(\lambda)} e^{i\xi_p} + a_{-\mathbf{p}}^{\dagger(\lambda)} e^{-i\xi_p} \right)$ where ξ_p is the phase of \mathcal{A}_p . They are restored after eq. (3.11).

This expression is a general result.

Since we consider the case where the electromagnetic fields has a peak strength at $p_B \geq 1\text{Mpc}^{-1}$ that are much smaller than the Planck pivot scale $k = 0.05\text{Mpc}^{-1}$, the delta function $\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k})$ in the integration in terms of \mathbf{p} and \mathbf{p}' can be approximated by $\delta(\mathbf{p}-\mathbf{p}')$. Performing the p' integral with $\delta(\mathbf{p}-\mathbf{p}')$, eq. (3.16) reads

$$\mathcal{P}_\zeta^{\text{em}}(k) > \frac{k^3}{2^2\pi^5} \int dN dN' \frac{1}{\epsilon \rho_{\text{inf}}} \frac{1}{\epsilon \rho_{\text{inf}}} \int^k d^3p \frac{|\partial_\eta \mathcal{A}_p(\eta)|^2}{a^4(\eta)} \frac{|\partial_{\eta'} \mathcal{A}_p(\eta')|^2}{a^4(\eta')}. \quad (3.17)$$

By using eq. (3.8), we finally obtain

$$\mathcal{P}_\zeta^{\text{em}}(k) > \frac{k^3}{4\pi} \int dN dN' \frac{1}{\epsilon \rho_{\text{inf}}(\eta)} \frac{1}{\epsilon \rho_{\text{inf}}(\eta')} \int^k \frac{d^3p}{p^6} \mathcal{P}_E(\eta, p) \mathcal{P}_E(\eta', p). \quad (3.18)$$

In the following discussion, we investigate the constraint on the inflationary magnetogenesis based on the above expression with the observed lower bound for the magnetic field given by eq. (2.1).

4 Model Independent Constraint

In this section, we discuss the condition that the induced curvature power spectrum eq. (3.18) does not exceed the observed value. That condition leads to a general and critical constraint on the inflationary magnetogenesis scenarios.

To evaluate eq. (3.18), we adopt the strategy outlined in sec. 2.3. In eq. (3.18), the interval of the N integral should be performed from the end of inflation to the time when the electric field is produced. In the standard inflationary magnetogenesis models, the electric field is initially produced when the scale of interest exits the horizon and evolves until the end of inflation. Then the integration interval should be $N = [0, \ln |k\eta_f|^{-1}]$ where k is the scale of interest and $N = \ln |k\eta_f|^{-1}$ denotes a time at the exit of the horizon. However, the time dependence of the electric field from the initial time to the end of inflation is quite dependent on what model is considered. Hence, as we have discussed in sec. 2.3, to obtain the conservative constraint in a model independent way, we consider only the integration during last 1 e-folds $N = [0, 1]$ and assume that the vector potential $\mathcal{A}_k(\eta)$ is a simple power-law during that period. Moreover, we consider that the power spectrum of the electric field has a peak at a wavenumber p_B which is related to the observed magnetic fields as shown in eq. (2.1). That is, we assume the mode function $\mathcal{A}_k(\eta)$ as

$$\mathcal{A}_k(\eta) = \left(\frac{\eta}{\eta_f}\right)^{-n} \mathcal{A}_k(\eta_f), \quad (e\eta_f \leq \eta \leq \eta_f, \ k \sim p_B), \quad (4.1)$$

and by substituting eq. (4.1) into eq (3.8) we can relate the time dependent power spectrum of the electric field to that of the magnetic field at the end of inflation as

$$\mathcal{P}_E(\eta, k) = \frac{n^2}{k^2\eta^2} \mathcal{P}_B(\eta, k) = \frac{n^2}{k^2\eta^2} \left(\frac{\eta}{\eta_f}\right)^{4-2n} \mathcal{P}_B(\eta_f, k), \quad (e\eta_f \leq \eta \leq \eta_f, \ k \sim p_B). \quad (4.2)$$

To connect the magnetic field at the end of inflation, η_f , and the present value, we assume that no amplification of the magnetic field occurs and hence it dilutes adiabatically after inflation, $\mathcal{P}_B \propto a^{-4}$. Although the magnetic fields on small scales vanish until today due to

the plasma dissipation effect, such dissipation scale is about 1 AU which is much smaller than the scale of interest here and then the adiabatic dilution should be valid [46]. For simplicity, we also assume the instantaneous reheating.¹⁰ Then $\mathcal{P}_B(\eta_f, k)$ is directly connected with the present $\mathcal{P}_B(\eta_{\text{now}}, k)$ as

$$\mathcal{P}_B(\eta_f, k) = \frac{\rho_{\text{inf}}}{\rho_\gamma} \mathcal{P}_B(\eta_{\text{now}}, k), \quad (4.3)$$

where $\rho_\gamma \approx 5.2 \times 10^{-12} \text{G}^2$ is the present energy density of radiation. The lower bound for the strength of the magnetic field given by eq. (2.1) is rewritten in terms of the power spectrum as

$$\mathcal{P}_B(\eta_{\text{now}}, k) \gtrsim \mathcal{P}_B^{\text{obs}}(p_B) \equiv 10^{-30} \text{G}^2 \left(\frac{p_B}{1 \text{Mpc}^{-1}} \right) \quad \text{for } k \sim p_B \geq 1 \text{Mpc}^{-1}. \quad (4.4)$$

Substituting eqs. (4.2), (4.3) and (4.4) into eq. (3.18), the p integral in eq. (3.18) reads

$$\begin{aligned} \int \frac{d^3 p}{p^6} \mathcal{P}_E(\eta, p) \mathcal{P}_E(\eta', p) &= \left(\frac{\rho_{\text{inf}}}{\rho_\gamma} \right)^2 \frac{n^4}{\eta^2 \eta'^2} \left(\frac{\eta}{\eta_f} \right)^{4-2n} \left(\frac{\eta'}{\eta_f} \right)^{4-2n} \int \frac{d^3 p}{p^{10}} \mathcal{P}_B^2(\eta_{\text{now}}, p) \\ &\gtrsim 4\pi \left(\frac{\rho_{\text{inf}}}{\rho_\gamma} \right)^2 \frac{n^4}{\eta^2 \eta'^2} \left(\frac{\eta}{\eta_f} \right)^{4-2n} \left(\frac{\eta'}{\eta_f} \right)^{4-2n} \left(\mathcal{P}_B^{\text{obs}}(p_B) \right)^2 \frac{p_B^{-7}}{7} \end{aligned} \quad (4.5)$$

where $e\eta_f \leq \eta$, $\eta' \leq \eta_f$. In the second line of the above equation, an inequality comes from the assumption that $\mathcal{P}_B(\eta_{\text{now}}, p) \simeq \text{constant}$ in p for $p \sim p_B$ and $\mathcal{P}_B(\eta_{\text{now}}, p) \simeq 0$ for $p \gg p_B$ and $p \ll p_B$ while it may have a finite value (see the discussion below eq. (2.1)). Then, as we have discussed in sec. 2.3, N integral within $N = [0, 1]$ in eq. (3.18) can be calculated as

$$\eta_f^{2n-4} \int_0^1 dN \frac{\eta^{2-2n}}{\epsilon \rho_{\text{inf}}} > \rho_{\text{inf}}^{-1} \eta_f^{2n-4} \int_{\eta_f}^{\epsilon \eta_f} d\eta \eta^{1-2n} = \rho_{\text{inf}}^{-1} \eta_f^{-2} \frac{1 - e^{2-2n}}{2n-2}, \quad (4.6)$$

where an inequality comes from the fact that we have used $0 < \epsilon \leq 1$ and $dN = -aHd\eta \simeq \frac{1}{1-\epsilon} d\ln \eta > d\ln \eta$.¹¹ We have also assumed that the energy density of the inflaton ρ_{inf} does not significantly vary for the last 1 e-fold. Thus, we can obtain the conservative lower bound for the power spectrum of the curvature perturbations induced from the electromagnetic fields during inflation as

$$\mathcal{P}_\zeta^{\text{em}}(k) > \frac{1}{7} \left[n^2 \frac{1 - e^{2-2n}}{2n-2} \right]^2 \left(\frac{k}{p_B} \right)^3 e^{4N_B} \left(\frac{\mathcal{P}_B^{\text{obs}}}{\rho_\gamma} \right)^2, \quad (4.7)$$

where we define $|p_B \eta_f|^{-1} = e^{N_B}$ and N_B is the e-folding number measured between the end of inflation and a time when the p_B mode exits the horizon during inflation. N_B can be written in terms of the energy density of the inflaton ρ_{inf} and p_B as [29, 48]

$$N_B \geq 58.8 - \ln \left(\frac{p_B}{H_0} \right) + \ln \left(\frac{\rho_{\text{inf}}^{1/4}}{10^{15} \text{GeV}} \right), \quad (4.8)$$

¹⁰In appendix. A we relax this assumption for the reheating stage and show that the similar constraint on the reheating energy scale ρ_{reh} can be obtained.

¹¹The factor $(1 - e^{2-2n})/(2n-2)$ in eq. (4.6) should be replaced by 1 for $n = 1$.

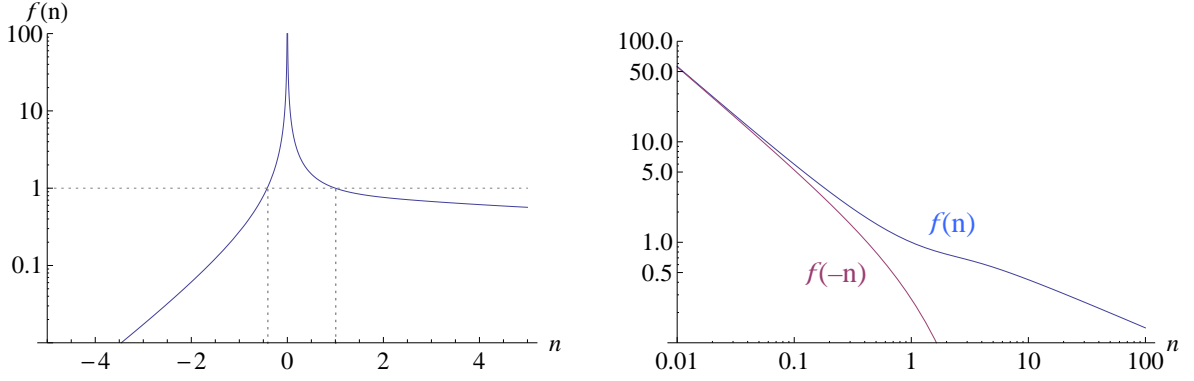


Figure 1. The behavior of $f(n)$ defined in eq. (4.11). The left panel is the log plot while the right panel is the log-log plot. It is shown that $f(n) = 1$ for $n = 1$ and $n \approx -0.42$. One can easily see that $f(n) \leq 1$ for $|n| \geq 1$ and $f(n) \gg 1$ only for $|n| \ll 1$.

where $H_0^{-1} = 4.4\text{Gpc}$ is the present horizon scale and we have assumed the instantaneous reheating, and then we have

$$\mathcal{P}_\zeta^{\text{em}}(k) > \frac{e^{4 \times 58.8}}{7} \left[n^2 \frac{1 - e^{2-2n}}{2n-2} \right]^2 \left(\frac{k}{p_B} \right)^3 \left(\frac{H_0}{p_B} \right)^4 \left(\frac{\mathcal{P}_B^{\text{obs}}}{\rho_\gamma} \right)^2 \left(\frac{\rho_{\text{inf}}^{1/4}}{10^{15}\text{GeV}} \right)^4. \quad (4.9)$$

Finally, by requiring that the induced curvature perturbations given by the above expression should not exceed the observed power spectrum $\mathcal{P}_\zeta^{\text{obs}}(k) = 2.2 \times 10^{-9}$ at the Planck pivot scale $k^{-1} = 20\text{Mpc}$ [49], we can obtain the upper bound on the inflationary energy scale as

$$\rho_{\text{inf}}^{1/4} < 30\text{GeV} \left(n^2 \frac{1 - e^{2-2n}}{2n-2} \right)^{-1/2} \left(\frac{p_B}{1\text{Mpc}^{-1}} \right)^{5/4} \left(\frac{B_{\text{obs}}}{10^{-15}\text{G}} \right)^{-1}. \quad (4.10)$$

Here, we use B_{obs} given by $\mathcal{P}_B^{\text{obs}} = B_{\text{obs}}^2 (p_B/1\text{Mpc}^{-1})$ for $p_B \geq 1\text{Mpc}^{-1}$ which is the strength of the magnetic field measured by blazar observations, as shown in sec. 2.1. The result eq. (4.10) depends on the parameter n in the factor $f(n)$ defined by

$$f(n) \equiv \left(n^2 \frac{1 - e^{2-2n}}{2n-2} \right)^{-1/2}. \quad (4.11)$$

$f(n)$ is plotted in fig.1 as a function of n . In this figure, one can see $f(n) \leq 1$ for $|n| \geq 1$. Therefore $f(n)$ can be roughly replaced by 1 in eq. (4.10) in the case of $|n| \geq 1$ and we obtain

$$\rho_{\text{inf}}^{1/4} < 30\text{GeV} \left(\frac{p_B}{1\text{Mpc}^{-1}} \right)^{5/4} \left(\frac{B_{\text{obs}}}{10^{-15}\text{G}} \right)^{-1}, \quad (|n| \geq 1). \quad (4.12)$$

This is a main conclusion of this paper.

As for the case with $|n| \ll 1$, namely $\mathcal{A}_p \simeq \text{const}$, the constraint eq. (4.10) seems to be relaxed because the electric field, $E \propto \partial_\eta \mathcal{A}_p$, becomes very small. Nevertheless, for $|n| \ll 1$, we can obtain a tighter constraint than eq. (4.12) by the following argument. This argument is based on the discussion that in order to achieve effective inflationary magnetogenesis there must exist a time when $n \sim \mathcal{O}(1)$ during inflation even if $|n| \ll 1$ for the last one e-fold, as we have mentioned in the last part of section 2.3.

For the last 1 e-folding time of inflation, the magnetic power spectrum behaves as $\mathcal{P}_B \propto a^{2n-4}$ (see eqs. (3.8) and (2.6)). Thus \mathcal{P}_B decreases in proportion to a^{-4} for $|n| \ll 1$, in other words, \mathcal{P}_B becomes much larger as goes back in time during inflation. On the other hand, to realize the effective production of the magnetic field during inflation, \mathcal{P}_B must significantly increase and hence n should reach $\mathcal{O}(1)$ at some e-folding time N_c . Let us estimate the induced $\mathcal{P}_\zeta^{\text{em}}$ generated within $N = [N_c, N_c + 1]$ by assuming that $\mathcal{A}_k(\eta)$ is well approximated as

$$\mathcal{A}_k(\eta) = \left(\frac{\eta}{\eta_c}\right)^{-n} \mathcal{A}_k(\eta_c), \quad (e\eta_c \leq \eta \leq \eta_c, k \sim p_B), \quad (4.13)$$

where $\eta_c \equiv e^{N_c} \eta_f$. In such case, the p integral in eq. (3.18) reads

$$\begin{aligned} & \int \frac{d^3 p}{p^6} \mathcal{P}_E(p, N) \mathcal{P}_E(p, N') \\ &= \left(\frac{\rho_{\text{inf}}}{\rho_\gamma}\right)^2 \frac{n^4}{\eta^2 \eta'^2} \left(\frac{\eta}{\eta_c}\right)^{4-2n} \left(\frac{\eta'}{\eta_c}\right)^{4-2n} \int \frac{d^3 p}{p^{10}} e^{8N_c} \mathcal{P}_B^2(p, \eta_{\text{now}}) \\ &\gtrsim 4\pi \left(\frac{\rho_{\text{inf}}}{\rho_\gamma}\right)^2 \frac{n^4}{\eta^2 \eta'^2} \left(\frac{\eta}{\eta_c}\right)^{4-2n} \left(\frac{\eta'}{\eta_c}\right)^{4-2n} \left(e^{4N_c} \mathcal{P}_B^{\text{obs}}(p_B)\right)^2 \frac{p_B^{-7}}{7}. \end{aligned} \quad (4.14)$$

This equation looks similar to eq. (4.5). However, note that since $\mathcal{P}_B \propto a^{-4}$ for $N = [0, N_c]$, the required strength of the magnetic field becomes large as $\mathcal{P}_B(p_B, \eta_c) = e^{4N_c} \mathcal{P}_B(p_B, \eta_f)$ at N_c . The time integration in eq. (3.18) is given by

$$\eta_c^{2n-4} \int_{N_c}^{N_c+1} dN \eta^{2-2n} = \eta_c^{2n-4} \int_{\eta_c}^{e\eta_c} d\eta \eta^{1-2n} = \eta_c^{-2} \frac{1 - e^{2-2n}}{2n-2}. \quad (4.15)$$

In addition, the slow-roll parameter ϵ is much smaller than unity because N_c is taken to be a some time during inflation. Thus, $\mathcal{P}_\zeta^{\text{em}}(k, \eta_c)$ is bounded as

$$\mathcal{P}_\zeta^{\text{em}}(k, \eta_c) > \frac{1}{7} \left[n^2 \frac{1 - e^{2-2n}}{2n-2} \right]^2 \left(\frac{k}{p_B}\right)^3 \left(\frac{\mathcal{P}_B^{\text{obs}}}{\rho_\gamma}\right)^2 e^{4N_B} \times \left(\frac{e^{4N_c}}{\epsilon^2}\right), \quad (4.16)$$

where we use $e^{4N_c}/(p_B \eta_c)^4 = e^{4N_B}$. Note that except for the last factor, $e^{4N_c}/\epsilon^2 \gg 1$, this equation is same as eq. (4.7). As a result, the constraint on $\rho_{\text{inf}}^{1/4}$ becomes tighter by $\sqrt{\epsilon} e^{-N_c}$ than eq. (4.12) in cases where $|n| \ll 1$ for the last one e-fold of inflation, as

$$\rho_{\text{inf}}^{1/4} < 30 \text{ GeV} \left(\frac{p_B}{1 \text{ Mpc}^{-1}}\right)^{5/4} \left(\frac{B_{\text{obs}}}{10^{-15} \text{ G}}\right)^{-1} \sqrt{\epsilon} e^{-N_c}, \quad (|n| \ll 1). \quad (4.17)$$

The reason why the stronger constraint is obtained can be understood as follows. If the vector potential \mathcal{A}_p stops growing and becomes constant during inflation ($n \sim 0$), the electric field becomes negligible. But, at the same time, the magnetic field begins to rapidly decrease, $B \propto a^{-2}$. To achieve the sufficient magnetic production, much stronger magnetic field should be generated before \mathcal{A}_p stops. Therefore the induced curvature perturbation that are generated right before \mathcal{A}_p stops is larger than the case with $|n| \geq 1$.¹²

¹²On the other hand, right before \mathcal{A}_p stops, the physical wave length of the mode p is smaller than that at the end of inflation. Thus the hierarchy between the electric field and the magnetic field is milder (see eq. (2.5)). Although this effect somewhat weakens the constraint, the bound on ρ_{inf} becomes tighter than eq. (4.12), as a result.

Consequently, we conclude that eq. (4.12) holds as a conservative and general constraint on inflationary magnetogenesis for any n :

$$\rho_{\text{inf}}^{1/4} < 30\text{GeV} \left(\frac{p_B}{1\text{Mpc}^{-1}} \right)^{5/4} \left(\frac{B_{\text{obs}}}{10^{-15}\text{G}} \right)^{-1}, \quad (p_B \geq 1\text{Mpc}^{-1}). \quad (4.18)$$

5 Summary and discussion

In this paper, we show that inflationary magnetogenesis is generally constrained as eq. (4.18) by requiring that the curvature perturbation induced by the electric field during inflation should be smaller than the Planck observation value: $\mathcal{P}_{\zeta}^{\text{obs}}(k) = 2.2 \times 10^{-9}$. We emphasize that our argument is model independent as we outlined in sec. 2.3. The main result eq. (4.18) indicates that inflationary magnetogenesis is under pressure in several ways.

First, it is known that the reheating (thermalization) energy scale is bounded as $\rho_{\text{reh}}^{1/4} \gtrsim 10\text{MeV}$ in order to achieve a successful BBN [50]. Therefore even if eq. (4.18) is almost saturated, for example $\rho_{\text{inf}}^{1/4} \sim 10\text{GeV}$, the reheating should be quickly completed.

Second, the generation of the observed curvature perturbation is in danger. Eq. (4.18) can be translated as

$$H_{\text{inf}} < 2 \times 10^{-7}\text{eV} \left(\frac{p_B}{1\text{Mpc}^{-1}} \right)^{5/2} \left(\frac{B_{\text{obs}}}{10^{-15}\text{G}} \right)^{-2}, \quad (5.1)$$

where H_{inf} is the Hubble parameter during inflation. For a scalar field to acquire a perturbation during inflation, its mass should be smaller than H_{inf} . Thus inflaton field or a spectator field which is responsible to produce $\mathcal{P}_{\zeta}^{\text{obs}}$ must be extremely light during inflation. During reheating era, however, it has to quickly decay into the standard model particles to cause the BBN properly. Furthermore, in the case of single slow roll inflation, eq.(5.1) and the COBE normalization indicate an extreme slow-roll $\epsilon < 4 \times 10^{-62}$ which demands a dedicated inflation model. It is interesting to note that eq.(5.1) corresponds to the very small tensor-to-scalar ratio, $r < 7 \times 10^{-61}$. Hence a detection of background gravitational waves in the future excludes inflationary magnetogenesis. ¹³

Third, in such a low reheating temperature, thermal production of the dark matter or the baryon number seems hopeless. Since 30GeV is accessible by colliders, effects beyond the standard model have been severely restricted. To realize the dark matter production and baryogenesis, a non-thermal mechanism like the direct decay of inflaton should be considered.

In spite of these negative implications, since we have the observational evidence of the magnetic fields in the universe and we are lack of a plausible magnetogenesis model, the inflationary origin of the magnetic field is still an appealing idea. It should be noted that we assume no amplification of the magnetic fields after inflation to derive eq. (4.18). Thus our result might imply that inflationary magnetogenesis need an additional amplification or a non-adiabatic dilution of magnetic fields after inflation. If the magnetic field generated during inflation is amplified by some mechanism like preheating process [52] or the inverse cascade [53, 54], the constraint is alleviated.

Another possible way out from our constraint is to produce a large amplitude of the vector potential before the horizon crossing. It is known that, in the so-called strong coupling regime of the kinetic coupling model, the electric field is not much stronger than the magnetic

¹³See ref. [48] in which our model-independent constraint is followed up in the light of the BICEP2 result [51].

field and the backreaction and curvature perturbation problems are evaded (if loop effects are neglected) [30]. This is because the vector potential \mathcal{A}_k is almost constant on super-horizon ($n \simeq 0$ in our language). The magnetic field is produced since \mathcal{A}_k has a large amplitude at the horizon crossing due to the small kinetic function. However, as discussed below eq. (3.4), such model suffers from the strong coupling problem and reliable calculations are difficult to be done. If a large amplitude of a static vector potential is realized without the strong coupling or one can take into account the loop effects in some non-perturbative way, sufficient magnetogenesis might be achieved.

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A Non instantaneous reheating

In this appendix, we relax the assumption of the instantaneous reheating. First, it is useful to introduce the reheating parameter [55, 56]:

$$R \equiv \left(\frac{a_f}{a_{\text{reh}}} \right) \left(\frac{\rho_{\text{inf}}}{\rho_{\text{reh}}} \right)^{1/4} = \left(\frac{a_{\text{reh}}}{a_f} \right)^{\frac{1-3\bar{w}}{4}} = \left(\frac{\rho_{\text{reh}}}{\rho_{\text{inf}}} \right)^{\frac{1-3\bar{w}}{12(1+\bar{w})}}, \quad (\text{A.1})$$

where subscript “reh” denotes the end of reheating (thermalization) and \bar{w} is the effective equation of state parameter that is the averaged w over the intermediate era between the end of inflation and the end of thermalization. When the assumption of the instantaneous reheating is relaxed, two equations in sec. 4 are modified. One is eq. (4.3) which should be modified as

$$\mathcal{P}_B(p, \eta_f) = R^{-4} \frac{\rho_{\text{inf}}}{\rho_\gamma} \mathcal{P}_B(p, \eta_{\text{now}}). \quad (\text{A.2})$$

The other is eq. (4.8) and it is changed as

$$N_B = 58.8 - \ln \left(\frac{p_B}{H_0} \right) + \ln \left(\frac{\rho_{\text{inf}}^{1/4}}{10^{15} \text{GeV}} \right) + \ln R. \quad (\text{A.3})$$

Therefore the generalization to non-instantaneous reheating cases can be taken into account by multiplying the right hand side of eq. (4.18) by R . If $\bar{w} > 1/3$ and $R > 1$, the constraint on ρ_{inf} becomes milder because the dominant component of the energy density decays faster than the magnetic fields.

Nevertheless, it is important that $\rho_{\text{reh}}^{1/4}$ can not be bigger than the upper bound on $\rho_{\text{inf}}^{1/4}$ of the instantaneous reheating case, namely eq. (4.18). Since eq. (A.1) reads $\rho_{\text{reh}}^{1/4} = R^{\frac{3(1+\bar{w})}{1-3\bar{w}}} \rho_{\text{inf}}^{1/4}$, $\rho_{\text{reh}}^{1/4}$ can not exceed $R^{\frac{4}{1-3\bar{w}}} \times (\text{r.h.s of eq. (4.18)})$. On the other hand, ρ_{reh} is smaller than ρ_{inf} , by definition. Except for $\bar{w} = 1/3$, the constraint on $\rho_{\text{reh}}^{1/4}$ can be written as

$$\rho_{\text{reh}}^{1/4} < \begin{cases} R^{\frac{4}{1-3\bar{w}}} \times 30\gamma \text{GeV} & (\bar{w} > 1/3, R^{\frac{4}{1-3\bar{w}}} < 1) \\ \rho_{\text{inf}}^{1/4} < R \times 30\gamma \text{GeV} & (\bar{w} < 1/3, R < 1) \end{cases}, \quad (\text{A.4})$$

where $\gamma \equiv \left(\frac{p_B}{1\text{Mpc}^{-1}}\right)^{5/4} \left(\frac{B_{\text{obs}}}{10^{-15}\text{G}}\right)^{-1}$. Therefore the reheating (thermalization) energy scale ρ_{reh} is maximized in the instantaneous reheating where $R = 1$ and $\rho_{\text{inf}} = \rho_{\text{reh}}$.

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